

RAMSEY NUMBERS OF CONNECTED CLIQUE MATCHINGS

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ABSTRACT. We determine the Ramsey number of a connected clique matching. That is, we show that if G is a 2-edge-coloured complete graph on $(r^2 - r - 1)n - r + 1$ vertices, then there is a monochromatic connected subgraph containing n disjoint copies of K_r , and that this number of vertices cannot be reduced.

1. INTRODUCTION

For a graph G , the Ramsey number $R(G)$ is defined to be the smallest integer N such that every 2-colouring of the edges of the complete graph on N vertices contains a monochromatic subgraph isomorphic to G . The most fundamental problem in Ramsey theory is determining the order of magnitude of the Ramsey numbers of cliques. An exponential upper bound was given by Erdős and Szekeres [7], and in an early use of the probabilistic method Erdős [6] gave an exponential lower bound. In spite of progress being made on both upper and lower bounds (see [5, 10]) even the size of the exponent is not known asymptotically. It becomes easier however to look for multiple copies of the cliques.

The Ramsey numbers of multiple copies of graphs were studied in [4] by Burr, Erdős, and Spencer who determined the Ramsey number of nK_3 exactly and of multiple copies of a general graph G up to a constant depending only on G . In particular they showed $R(nK_3) = 5n$ and for $r \geq 4$ that $(2r - 1)n - 1 \leq R(nK_r) \leq (2r - 1)n + C_r$, determining the Ramsey number of a K_r -matching up to a constant.

The aim of this note is to add a connectivity requirement, studying the Ramsey numbers of connected copies of cliques. Although not technically a Ramsey number by the definition given above, we let $R(c(nH))$ denote the least N such that every 2-colouring of the edges of K_N contains a monochromatic copy of nH in a connected component of the same colour. This was first studied by Gyárfás and Sárközy [9] who solved the problem for $H = K_3$ showing $R(c(nK_3)) = 7n - 2$. We solve the problem for all larger cliques.

Theorem 1. *For $r \geq 4$ and $n \geq R(K_r)$ we have*

$$R(c(nK_r)) = (r^2 - r + 1)n - r + 1.$$

The connectivity requirement here proves to be very significant as for large values of r the Ramsey number of a connected K_r -matching is approximately $r/2$ times larger than that of a standard K_r -matching.

The lower bound in Theorem 1 is a special case of the following observation of Burr [3].

Proposition 2. *For every connected graph G containing at least one edge*

$$R(G) \geq (\chi(G) - 1)(|G| - 1) + \sigma(G)$$

where $\sigma(G)$ is the smallest size of a colour class over all $\chi(G)$ -proper-colourings of G .

To see this partition the vertices of K_N , with $N = (\chi(G) - 1)(|G| - 1) + \sigma(G) - 1$, into parts $X_1, \dots, X_{\chi(G)-1}$ and Y such that $|Y| = \sigma(G) - 1$ and $|X_i| = |G| - 1$ for each i . Colour all edges within each part with blue and colour all edges between different parts with red. There cannot be a blue copy of G since all connected components of blue are too small. There cannot be a copy of G in red since the red edges form a $\chi(G)$ -partite graph where the smallest part is too small.

This construction led Burr to conjecture that for any $\Delta, k \in \mathbb{N}$ there exists an n_0 such that any connected graph G on $n \geq n_0$ vertices with chromatic number k and maximum degree at most Δ satisfies $R(G) = (k - 1)(n - 1) + \sigma(G)$. Burr's conjecture was proven to be false by Graham, Rödl, and Rucinski [8]. Our result shows that Burr's conjecture does hold in the case of connected copies of cliques. We conjecture the following

Conjecture 3. *For any graph H there exists $n_0(H)$ such that for all $n \geq n_0$ we have*

$$R(c(nH)) = (\chi(H) - 1)(n|H| - 1) + n\sigma(H).$$

The lower bound holds by Proposition 2.

In [9] Gyárfás and Sárközy used their method for determining $R(c(nK_3))$ along with the Regularity Lemma to estimate asymptotically the Ramsey number of a C_n^2 that is missing a constant number of edges. We suspect, but have not checked all technicalities, that we could obtain a similar result, showing that for any r and any $\varepsilon > 0$ there exists c_r such that any 2-colouring of K_N with $N \geq (1 + \varepsilon)(r^2 - r + 1)n$ contains a monochromatic copy of a C_{rn}^{r-1} missing some c_r edges.

There is another stronger connectivity requirement on disjoint cliques for which it is interesting to study the Ramsey numbers. Suppose P and Q are copies of K_r . We say they are K_r -connected if there exist copies C_1, \dots, C_t of K_r such that in the sequence P, C_1, \dots, C_t, Q , consecutive copies of K_r share $r - 1$ vertices. A construction of Allen, Brightwell, and Skokan in [1] shows that if we wish to find a monochromatic set of n disjoint copies of K_r which are K_r -connected in K_N we need $N \geq nr^2 - 2r + 2$. This construction was given as a lower bound for the Ramsey number $R(C_{rn}^{r-1})$ where C_m^k is the m -th power of a cycle, obtained by joining all vertices of C_m at distance at most k . In [2], with Allen and Skokan we determine the Ramsey number $R(C_m^2)$ for large m via studying the Ramsey numbers of K_3 -connected copies of disjoint triangles.

2. PROOF OF THEOREM 1

The lower bound for Theorem 1 follows from the construction of Burr given in the previous section. Before giving a proof of the upper bound we sketch the main ideas.

For $r \geq 4$ and $n \geq R(K_r)$, consider a 2-colouring of $G = K_N$ with $N = (r^2 - r + 1)n - r + 1$. Since in any 2-colouring of a complete graph one of the colours is connected, we assume G is connected in red and look for either a red copy of nK_r or a blue connected copy of nK_r .

We then show that G can be partitioned into a maximum set of disjoint red copies of K_r and a set of $r - 1$ large blue cliques such that between any two of the blue cliques all edges are red. We call these blue cliques B_1, \dots, B_{r-1} , and the union of their vertex sets B . We let \mathcal{R} denote the maximum set of red copies of K_r , and we let R denote the vertex set of \mathcal{R} . We then consider edges between the red copies of K_r in \mathcal{R} and the sets B_1, \dots, B_{r-1} . We show that each clique of \mathcal{R} is of one of two types (see claim 4) with regards to how the edges between that clique and B are coloured. Furthermore each type gives a way of assigning vertices of the red K_r to some B_i such that almost all of the edges between the assigned vertex and B_i are blue. For each $i = 1, \dots, r - 1$, we let D_i denote the union of B_i along with the vertices of R that were assigned to B_i . We then use an averaging argument to show that there exists an i such that $|D_i| \geq rn$ and we look for a blue nK_r on this D_i . Since B_i was a blue clique and vertices assigned to B_i were connected to B_i in blue, this nK_r is connected in blue. If more than $(r - 1)n$ of the vertices of D_i came from B_i we can find the nK_r greedily. Otherwise we use more information coming from the types of red K_r vertices assigned to B_i came from to find an nK_r . There is one special case for which this method fails. If this occurs we find a connected nK_r in a different $D_{i'}$.

Proof of Theorem 1. For $r \geq 4$ and $n \geq R(K_r)$, let $N = (r^2 - r + 1)n - r + 1$ and consider a red/blue edge colouring of $G = K_N$. Any 2-colouring of K_N is connected in one of the colour, since if Blue has more than one connected component, Red contains a complete multi-partite graph between blue components which connects the graph in red. Without loss of generality we assume G is connected in red. Consider a maximal set of vertex disjoint red copies of K_r . We call this set of cliques \mathcal{R} and its vertex set R . Note that $|R| \leq r(n - 1)$, otherwise we would be done. Greedily looking for vertex disjoint blue copies of K_r on the rest of the graph results in covering all but at most $R(K_r) - 1$ vertices of the graph with monochromatic copies of K_r . Call this set of uncovered vertices Z .

Let B denote the vertices covered by these blue cliques. We next partition B into blue connected components.

Between components all edges are red, so there cannot be more than $r - 1$ such components or we would have a new red K_r to add to \mathcal{R} . This would contradict the maximality of \mathcal{R} . There also cannot be fewer than $r - 1$ such components, or, since each component is no larger than $r(n - 1)$, we would have

$$|B| + |R| + |Z| \leq (r - 2)r(n - 1) + r(n - 1) + R(K_r) - 1 = (r^2 - r)n - r^2 + r - 1 + R(K_r),$$

which is a contradiction, since for $n > R(K_r) - r^2 + 2r - 2$ this is less than N . Therefore there must be exactly $r - 1$ blue components and we call them B'_1, \dots, B'_{r-1} . If any of these components contained a red edge we would have another red K_r to add to \mathcal{R} , and so each B'_i is a blue clique. Since the blue components are disconnected, each vertex in Z can be adjacent in blue to vertices of at most one B'_i . If any vertex in Z had a red neighbour in each B'_i we would have a new red K_r to add to \mathcal{R} . Therefore all vertices in Z are adjacent to $r - 2$ of the B'_i entirely in red and the remaining blue component entirely in blue. For each $i = 1, \dots, r - 1$, form B_i by adding to B'_i the vertices of Z that are adjacent in blue to all of B'_i . Note that there cannot be any red edges in any B_i or we would have a new

red K_r to add to \mathcal{R} and also between any two distinct B_i all edges are red since they are not blue connected.

We now consider the colour of edges between the cliques of \mathcal{R} and the sets B_1, \dots, B_{r-1} . First, recalling that all vertices in R are adjacent in blue to vertices in at most one B_i , we say a vertex of R is *paired with* B_i if it is adjacent in red to all B_j for all $j \neq i$. The following claim identifies two possible properties of cliques of \mathcal{R} .

Claim 4. *Let C be a red K_r from \mathcal{R} . Then one of the following holds:*

- (i) *For each B_i there is a vertex in C adjacent in blue to all but at most one vertex of B_i .*
- (ii) *For all but two values of i there is exactly one vertex of C which is adjacent in blue to all vertices of B_i . Furthermore there is a j such that the three remaining vertices of C are adjacent to all of B_j .*

Proof. Consider some $C \in \mathcal{R}$. Each vertex in C can have blue neighbours in at most one B_i and so is entirely adjacent in red to the remaining $r - 2$ blue components.

Suppose firstly that some $c_i \in C$ is paired with B_i but also has at least one neighbour in red in B_i . Then if two other vertices $c_j, c'_j \in C$ were paired with the same B_j , we could break up C to make two new red copies of K_r , contradicting the maximality of \mathcal{R} . One of these red cliques uses c_i , its red neighbour in B_i and vertices of B_k for $k \neq i$. The other uses c_j, c'_j and vertices of each B_k for $k \neq j$. Therefore, if such a $c_i \in C$ exists, all remaining vertices of C are paired with distinct components B_j . Furthermore, if any of them has more than one red neighbour in the set they are paired with, we could make two new red copies of K_r , again contradicting the maximality of \mathcal{R} . One of these would use c_i as before, and the other would use the other vertex with red neighbours in the set it is paired with. Thus, if c_i as above exists, (i) holds.

Secondly, suppose all vertices in C are adjacent entirely in blue to the set they are paired with. There cannot be two vertices c_i, c'_i paired with some B_i and another two vertices c_j, c'_j paired with some B_j . If there were, we would create two new red copies of K_r using c_i, c'_i and a vertex from each B_k with $k \neq i$ for one, and c_j, c'_j and a vertex from each B_k with $k \neq j$ for the other. This would again break the maximality of \mathcal{R} . There also cannot be four or more vertices paired with the same B_i , or we could use two pairs of them to make two new red copies of K_r along with vertices from B_k for $k \neq i$. Therefore we either have some B_i with two vertices of C paired with it and every other B_j has one vertex paired with them, as in (i), or we have some B_i with three vertices paired with it and all but one of the remaining components B_j have one vertex paired with them, as in (ii). \square

The next claim tells us that if we have two sets of three vertices from cliques of \mathcal{R} that are paired with the same B_i , then the edges between these two sets are all blue.

Claim 5. *Let C and C' be cliques of \mathcal{R} with vertices $x_1, x_2, x_3 \in C$ and $y_1, y_2, y_3 \in C'$ all paired with the same B_i . Then all the edges $x_j y_k$ for $j, k \in \{1, 2, 3\}$ are blue.*

Proof. Suppose for contradiction and without loss of generality that $x_1 y_1$ is red. Then we could create three new red copies of K_r at the cost of C and C' , contradicting the

maximality of \mathcal{R} . These three copies of K_r would use the pairs of vertices $\{x_1, y_1\}$, $\{x_2, x_3\}$ and $\{y_2, y_3\}$ along with vertices from B_j for $j \neq i$. \square

We now use Claim 4 to partition the red cliques of \mathcal{R} depending on which option of the claim they satisfy. Let $S \subseteq \mathcal{R}$ be the set of cliques satisfying (i), and $T \subseteq \mathcal{R}$ be the set of cliques satisfying (ii). For each $C \in S$ and each B_i there is at least one vertex of C which is adjacent in blue to all but at most one vertex of B_i . For each i , construct S_i by selecting one such vertex from each $C \in S$. For each $C \in T$, all vertices are adjacent entirely in blue to exactly one B_i . For each i , construct T_i by taking the vertices of each clique of T that are entirely adjacent in blue to B_i . Let $D_i = B_i \cup S_i \cup T_i$. We further split up T_i into sets T_i^* and T_i^Δ , depending on whether one or three vertices from that red K_r were added to T_i . Given a vertex $u \in T_i$ let C be the clique of T containing u . If u is the only vertex of C belonging to T_i then u belongs to T_i^* . Otherwise three vertices of C must belong to T_i in which case all three belong to T_i^Δ . Observe that $|T_i| = |T_i^*| + |T_i^\Delta|$ and $n - |S_i| - |T_i^*| \geq \frac{1}{3}|T_i^\Delta| + 1$.

We shall find a blue connected copy of nK_r on a D_i such that either $|D_i| \geq rn + 1$, or $|D_i| \geq rn$ and $|T_i^\Delta| \neq 3$. We first proceed to show that such a D_i exists.

Because the D_i for $i = 1, \dots, r-1$ cover all vertices of the graph except for one from each clique of S , we have

$$N = (r^2 - r + 1)n - r + 1 = |S| + \sum_{i=1}^{r-1} |D_i|.$$

By averaging and using $|S| \leq n - 1$, there exists an i such that

$$|D_i| \geq \left(r + \frac{1}{r-1}\right)n - 1 - \frac{|S|}{r-1} \geq rn - 1 + \frac{1}{r-1}.$$

Since $|D_i|$ is an integer it is at least rn . Furthermore if $|S| \leq n - r$ it is at least $rn + 1$. Therefore if $|S| \leq n - r$ we can find a suitable D_i . If $|S|$ is larger than this, the next claim shows that either we still have a D_i with $|D_i| \geq rn + 1$ or we have at least two of size at least rn . In the latter case, we can find one of these with $|T_i^\Delta| \neq 3$.

Claim 6. *Suppose $|S| = n - \ell$ for some $1 \leq \ell \leq r - 1$. Then either there exist ℓ values of i for which $|D_i| \geq rn$ or there exists one value of i for which $|D_i| \geq rn + 1$.*

Proof. Suppose for contradiction that the $\ell - 1$ largest sets D_i all have at most rn vertices whilst all others have at most $rn - 1$. This gives

$$|S| + \sum_{i=1}^{r-1} |D_i| \leq n - \ell + (\ell - 1)rn + (r - \ell)(rn - 1) = (r^2 - r + 1)n - r < N$$

achieving a contradiction. \square

Since $|T| \leq n - 1 - |S|$, we see $|T| \leq \ell - 1$, and so if $|S| \geq n - r + 1$ we can find a value of i such that either $|D_i|$ is at least $rn + 1$ or $|D_i| \geq rn$ and $|T_i^\Delta| \neq 3$.

We begin using only the assumption $|D_i| \geq rn$. Then

$$|B_i| \geq rn - |S_i| - |T_i^*| - |T_i^\Delta|$$

and so $|B_i| - (r-1)(|S_i| + |T_i^*|) \geq r(n - |S_i| - |T_i^*|) - |T_i^\Delta| \geq (\frac{r}{3} - 1)|T_i^\Delta| + r \geq 1$. Since all vertices of $S_i \cup T_i^*$ are adjacent in blue to all but at most one vertex of B_i , we can extend all vertices of $S_i \cup T_i^*$ to disjoint blue copies of K_r using B_i . We now look to find $n - |S_i| - |T_i^*|$ disjoint blue copies K_r on the remaining vertices of B_i and T_i^Δ . Let \tilde{B}_i denote the remaining vertices of B_i , noting that $|\tilde{B}_i| \geq (\frac{r}{3} - 1)|T_i^\Delta| + r$. Recall that edges from T_i^Δ to B_i are blue and between different red triangles of T_i^Δ edges are blue.

If $|T_i^\Delta| = 0$, then since B_i is a blue clique we can find the remaining blue copies of K_r entirely on the rest of B_i .

Remembering that $|T_i^\Delta|$ is a multiple of three, suppose $|T_i^\Delta| \geq 6$. If $|T_i^\Delta| \geq 3r$, we first take blue copies of K_r on T_i^Δ such that the vertices of T_i^Δ that are not covered by these blue cliques consist of t red triangles with $2 \leq t \leq r-1$. We then cover the rest of T_i^Δ with blue copies of K_r by greedily taking one vertex from each of the t red triangles and extending this set of t vertices to a blue K_r using vertices from \tilde{B}_i . In this process three blue copies of K_r use both vertices of T_i^Δ and \tilde{B}_i , one for each vertex of the red triangles, and each such blue K_r used $r-t$ vertices from \tilde{B}_i . So long as $|\tilde{B}_i| \geq 3(r-t)$, this greedy procedure is successful. Since we have $|\tilde{B}_i| \geq (\frac{r}{3} - 1)|T_i^\Delta| + r$, we also have $|\tilde{B}_i| \geq 3t(\frac{r}{3} - 1) + r$, which is at least $3(r-t)$ for $t \geq 2$.

Finally, suppose $|T_i^\Delta| = 3$. Using Claim 6 we may assume $|D_i| \geq rn + 1$. We have that $|\tilde{B}_i| \geq rn + 1 - (r-1)(|S_i| + |T_i^*|) \geq (\frac{r}{3} - 1)|T_i^\Delta| + r + 1 = 2(r-1)$, and so we can extend two of the vertices of T_i^Δ to blue copies of K_r using \tilde{B}_i . If necessary we then cover the rest of \tilde{B}_i with more copies of K_r . This completes the proof. \square

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